Geometric approaches to amenability and unitarisability

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Introduction

In [1], Jacques Dixmier proved that any amenable group is unitarisable, and asked whether the converse holds. This question, still open, is known as the *Dixmier's unitarisability problem.* Since then, a lot of progress has been made towards the understanding of the class of unitarisable groups, especially by Gilles Pisier. The goal of the thesis is to explain another approach to one of Pisier's result, much more geometric than the initial one, developed by Schlicht in [2].

The cone $\mathcal{P}(\mathcal{H})$

We denote $\mathcal{P}(\mathcal{H})$ the set of positive invertible operators on a complex separable Hilbert space $\mathcal{H}$. It is a "cone" inside $\mathcal{B}(\mathcal{H})$.

Lemma. If $A, B \in \mathcal{P}(\mathcal{H})$ and $\lambda > 0$, then $A+B, \lambda A \in \mathcal{P}(\mathcal{H})$.

A metric structure on $\mathcal{P}(\mathcal{H})$

This cone carries a rich metric structure, different from the normed structure inherited from $\mathcal{B}(\mathcal{H})$, via the map

$$\begin{aligned} \mathrm{d}\colon \mathcal{P}(\mathcal{H}) \times \mathcal{P}(\mathcal{H}) &\longrightarrow [0,\infty) \\ (A,B) &\longmapsto \| \ln(A^{-1/2}BA^{-1/2}) \|. \end{aligned}$$

It is also a $\mathrm{Aut}(\mathcal{H})$−space via

$$\begin{aligned} \mathrm{Aut}(\mathcal{H}) \times \mathcal{P}(\mathcal{H}) &\longrightarrow \mathcal{P}(\mathcal{H}) \\ (A,P) &\longmapsto A \cdot P := APA^*. \end{aligned}$$

This action of $\mathrm{Aut}(\mathcal{H})$ is compatible with the distance d.

Proposition. $(\mathcal{P}(\mathcal{H}), \mathrm{d})$ is a metric space, and $\mathrm{Aut}(\mathcal{H})$ acts continuously, transitively and isometrically on $(\mathcal{P}(\mathcal{H}), \mathrm{d})$.

Lastly, this metric space is geodesic.

Proposition. Let $A, B \in \mathcal{P}(\mathcal{H})$. The map

$$\begin{aligned} \sigma(A,B,\cdot)\colon [0,1] &\longrightarrow \mathcal{P}(\mathcal{H}) \\ t &\longmapsto A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^t A^{1/2} \end{aligned}$$

is a geodesic between A and B. Moreover, $\mathrm{Aut}(\mathcal{H}) \curvearrowright \mathcal{P}(\mathcal{H})$ preserves those geodesics, *i.e.*

$$\sigma(A \cdot B, A \cdot C, t) = A \cdot \sigma(B, C, t)$$

for all $A \in \mathrm{Aut}(\mathcal{H})$, $B, C \in \mathcal{P}(\mathcal{H})$ and $t \in [0,1]$.

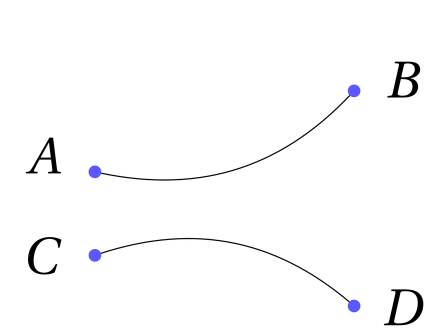

Figure 1: Two geodesics in $\mathcal{P}(\mathcal{H})$

The convexity inequality

A crucial feature of this structure is the convexity of the distance between two such geodesics. This reflects the non-positive curvature of $(\mathcal{P}(\mathcal{H}), \mathrm{d})$ (see Figure 1).

Theorem. Let $A, B, C, D \in \mathcal{P}(\mathcal{H})$. Then

$$\mathrm{d}(\sigma(A,B,t), \sigma(C,D,t)) \leq (1-t)\mathrm{d}(A,C) + t\mathrm{d}(B,D)$$

for all $t \in [0,1]$.

Towards Pisier's result

The class of unitarisable groups

Definition. A representation $\pi\colon G \longrightarrow \mathrm{Aut}(\mathcal{H})$ of G on a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is called unitarisable if there exists $S \in \mathrm{Aut}(\mathcal{H})$ so that $S^{-1}\pi(g)S$ is unitary, for all $g \in G$. In that case, we say S *unitarises* π. Moreover, π is called uniformly bounded if there exists $C > 0$ so that

$$\|\pi(g)\| \leq C$$

for all $g \in G$. For such representations, we set $|\pi| := \sup_{g \in G} \|\pi(g)\|$.

Definition. A group G is called unitarisable if all its uniformly bounded representations are unitarisable.

We then show that finite groups are unitarisable. More generally, we establish the major result of Dixmier in [1].

Theorem. Amenable groups are unitarisable. In particular, $\mathbb{Z}^d$, $\mathbb{Q}^d$, $\mathbb{R}^d$ are unitarisable for all $d \geq 1$.

We also provide examples of non-unitarisable groups.

Theorem. Any group containing F_2 as a subgroup is not unitarisable. In particular, F_d is not unitarisable for all $d \geq 2$, as well as $\mathrm{SL}_2(\mathbb{Z})$, $\mathrm{SL}_2(\mathbb{R})$, $\mathrm{GL}_2(\mathbb{Z})$, $\mathrm{GL}_2(\mathbb{R})$.

Group actions on $\mathcal{P}(\mathcal{H})$

A representation π of a group G on a Hilbert space $\mathcal{H}$ induces an action of the group on $\mathcal{P}(\mathcal{H})$, via

$$\begin{aligned} \theta_\pi\colon G \times \mathcal{P}(\mathcal{H}) &\longrightarrow \mathcal{P}(\mathcal{H}) \\ (g,P) &\longmapsto \pi(g)P\pi(g)^*. \end{aligned}$$

The crucial observation is that this action has fixed points if and only if π is unitarisable.

Lemma. If S unitarises π, then SS^* is a fixed point of θ_π. Conversely, if T is fixed by $\theta_\pi(g,\cdot)$ for all $g \in G$, then $T^{1/2}$ unitarises π.

Pisier's theorem takes then the following form.

Theorem. Let G be a unitarisable group.
There exist universal constants $\alpha, K > 0$, depending only on G, so that for any uniformly bounded representation π of G, there exists S unitarising π with

$$\|S\|\|S^{-1}\| \leq K|\pi|^\alpha.$$

Translated geometrically, this gives:

Theorem. Let G be a unitarisable group.
There exist universal constants $C \in \mathbb{R}$, $\alpha > 0$, depending only on G, so that for any uniformly bounded representation π of G, one has

$$\mathrm{d}(\mathcal{P}(\mathcal{H})^G, \mathrm{Id}_\mathcal{H}) = \mathrm{d}(\mathcal{P}(\mathcal{H})^G, O_{\mathrm{Id}_\mathcal{H}}) \leq C + \frac{\alpha}{2}\mathrm{diam}(O_{\mathrm{Id}_\mathcal{H}}).$$

References

[1] Jacques Dixmier, *Les moyennes invariantes dans les semi-groupes et leurs applications*, Acta Sci. Math. Szeged, vol. 12, 1950, 213–227.

[2] Peter Schlicht, *Amenable groups and a geometric view on unitarisability*, arXiv: 1502.01498, 2015.