Isoperimetric profiles and quantitative orbit equivalence for lampshufflers

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Abstract

These notes served as a basis for a joint talk given with Corentin Correia at the Autumn school *Probability and Dynamics on groups*, held at the University of Münster, in September 2025.

Contents

1 Isoperimetric profile and Følner function

Definition 1.1. Let $H = \langle S_H \rangle$ be a finitely generated group. Its *isoperimetric profile* is the function given by

$$\begin{aligned} j_{1,H} \colon \mathbb{N} &\longrightarrow \mathbb{R}_+ \\ n &\longmapsto \sup_{A \subset H;\ |A| \leqslant n} \frac{|A|}{|\partial_H A|} \end{aligned}$$

where $\partial_H A := \{y \in H \backslash A : \exists s \in S_H, \exists a \in A, y = as\}$ is the *boundary* of A in H.

- $f \preccurlyeq g$ if $\exists C > 0$, $f(n) \leqslant Cg(Cn)$ for all n large enough; and $f \simeq g$, if $f \preccurlyeq g$ and $g \preccurlyeq f$.
- $j_{1,H}$ is unbounded $\Longleftrightarrow$ H is amenable;
- Up to $\simeq$, $j_{1,H}$ is a QI invariant.

Note that a group is amenable if and only if its isoperimetric profile is unbounded. The idea to keep in mind is that the isoperimetric profile is a measurement of how much amenable a group is. The faster the isoperimetric profile tends to infinity, the more the group is amenable. It has now been computed for many classes of groups, among which:

- $j_{1,G}(n) \simeq n^{\frac{1}{d}}$ if G has polynomial growth of degree $d \geqslant 1$;
- $j_{1,G}(n) \simeq \ln(n)$ for solvable Baumslag-Solitar groups and lamplighters $F \wr \mathbb{Z}$, where F is a non-trivial finite group;

- $j_{1,G}(n) \simeq \ln(n)$ for any polycyclic group G with exponential growth [Pit95; Pit00], or more generally any exponential growth group G within the class GES of Tessera [Tes13, Corollary 5];
- $j_{1,F\wr N}(n) \simeq (\ln(n))^{\frac{1}{d}}$ with F finite and non-trivial, and N of growth degree $d \geqslant 1$ [Ers03].
- $j_{1,F\wr(F\wr\mathbb{Z})}(n) \simeq \ln(\ln(n))$ with F finite and non-trivial [Ers03].

The isoperimetric profile $j_{1,H}$ is the generalized inverse of the *Følner function* $\mathrm{Føl}_H$ of H, defined as

$$\begin{aligned} \mathrm{Føl}_H \colon \mathbb{N} &\longrightarrow \mathbb{R}_+ \\ n &\longmapsto \inf\left\{|A| : \frac{|\partial_H A|}{|A|} \leqslant \frac{1}{n}\right\}. \end{aligned}$$

Remark 1.2. Standard analysis arguments show that both $j_{1,H}$ and $\mathrm{Føl}_H$ can be extended to $\mathbb{R}_+$ and can be taken as *inverses* of each other; see e.g. [CD25, Remark 2.1].

In fact, [Ers03, Theorem 1] provides a general formula to compute the Følner function of a wreath product in term of Følner functions of its factors.

Theorem 1.3 ([Ers03])**.** *Let G and H be finitely generated groups. Suppose that*

$$(\star)\ \forall C > 0,\ \exists K > 0,\ C \cdot \mathit{Føl}_H(n) \leqslant \mathit{Føl}_H(Kn) \text{ for all } n \text{ large enough.}$$

Then

$$\mathit{Føl}_{G\wr H}(x) \simeq \mathit{Føl}_G(x)^{\mathit{Føl}_H(x)}.$$

The assumption $(\star)$ is easier to state with the isoperimetric profile: $j_{1,H}(Cx) = O(j_{1,H}(x))$ for all $C > 0$. In practice, it is satisfied for any group for which we know the isoperimetric profile.

Theorem 1.3 is a key ingredient for proving our main result.

2 Isoperimetric profiles of lampshufflers

Lampshufflers are constructed similarly to lamplighters. Fix a group H. Consider the group $\mathsf{FSym}(H)$ of all finitely supported bijections $H \longrightarrow H$. The group H acts naturally on $\mathsf{FSym}(H)$ by

$$(h \cdot \sigma)(x) := h\sigma(h^{-1}x)$$

for $h, x \in H$ and $\sigma \in \mathsf{FSym}(H)$. Indeed, if σ is finitely supported, then so is $h \cdot \sigma$ and $\mathrm{supp}(h \cdot \sigma) = h\mathrm{supp}(\sigma)$. Then the *lampshuffler group over* H is the semi-direct product

$$\mathsf{Shuffler}(H) := \mathsf{FSym}(H) \rtimes H.$$

Remarks:

- If H is finitely generated, then so is $\mathsf{Shuffler}(H)$;
- $\mathsf{Shuffler}(H)$ is amenable if and only if H is amenable.

Finally, let us sum-up here what is already known about the isoperimetric profile of lampshufflers:

Proposition 2.1 ([EZ21, Corollary 1.4]). *Let $H = \langle S_H \rangle$ be a finitely generated amenable group. Then*

$$F\o l_{\mathsf{Shuffler}(H)}(x) \geqslant \gamma_H(x)^{\gamma_H(x)}$$

where γ_H is the growth function of H with respect to S_H.

Inverting the inequality provides an upper bound on the isoperimetric profile of $\mathsf{Shuffler}(H)$, and in the case where $\gamma_H(x) \simeq x^d$, one gets

$$j_{1,\mathsf{Shuffler}(H)}(x) \preccurlyeq \left(\frac{\ln(x)}{\ln(\ln(x))}\right)^{\frac{1}{d}}$$

whereas if $\gamma_H(x) \simeq e^x$, one gets $j_{1,\mathsf{Shuffler}(H)}(x) \preccurlyeq \ln(\ln(x))$. Moreover, in the polynomial growth case, Erschler and Zheng proved that their upper bound is optimal, namely

$$j_{1,\mathsf{Shuffler}(H)}(x) \simeq \left(\frac{\ln(x)}{\ln(\ln(x))}\right)^{\frac{1}{d}}.$$

On the other hand, when H has exponential growth, the upper bound $\ln(\ln(x))$ is in general far from being optimal.

One of the main results of [CD25] will provide us sharp bounds.

Theorem 2.2 ([CD25, Theorem A]). *Let H be a finitely generated amenable group whose isoperimetric profile $j_{1,H}$ satisfies $(\star)$. Then one has*

$$j_{1,H}\left(\frac{\ln(x)}{\ln(\ln(x))}\right) \preccurlyeq j_{1,\mathsf{Shuffler}(H)}(x) \preccurlyeq j_{1,H}(\ln(x)).$$

In particular, note that if $j_{1,H}\left(\frac{\ln(x)}{\ln(\ln(x))}\right) \simeq j_{1,H}(\ln(x))$ $(\star\star)$, then one obtains the exact estimate

$$j_{1,\mathsf{Shuffler}(H)}(x) \simeq j_{1,H}(\ln(x)).$$

For instance, if $H = \mathrm{BS}(1,n)$, $n \geqslant 2$, we have $j_{1,H}(x) \simeq \ln(x)$ and thus $\mathsf{Shuffler}(\mathrm{BS}(1,n))$ has profile $\simeq \ln(\ln(x))$. This is a case where [EZ21, Corollary 1.4] is optimal.

On the other hand, $H = F \wr (F \wr \mathbb{Z})$ has exponential growth and profile $\simeq \ln(\ln(x))$, so that [EZ21, Corollary 1.4] only provides the upper bound $\ln(\ln(x))$, whereas Theorem 2.2 provides

$$j_{1,\mathsf{Shuffler}(F\wr(F\wr\mathbb{Z}))}(x) \simeq \ln(\ln(\ln(x))).$$

Remark 2.3. We emphasize here that, outside groups of polynomial growth, there are no examples of amenable finitely generated groups whose isoperimetric profile do not satisfy $(\star\star)$.

Iterated lampshufflers. Given a group H, we define inductively the *n-th iterated lampshuffler over H* as $\mathsf{Shuffler}^{\circ 0}(H) := H$ and $\mathsf{Shuffler}^{\circ n}(H) := \mathsf{Shuffler}(\mathsf{Shuffler}^{\circ(n-1)}(H))$.

One of the defect of Proposition 2.1 is that it does not to allow to compute precisely the profile for this class of groups. Indeed, if H has already exponential growth, so does $\mathsf{Shuffler}^{\circ n}(H)$ for any $n \geqslant 0$, so that we only get the upper bound

$$j_{1,\mathsf{Shuffler}^{\circ n}(H)}(x) \preccurlyeq \ln(\ln(x)),$$

independent of n.

On the other hand, the above upper and lower bounds can be used to derive estimates of iterated lampshufflers:

Theorem 2.4 ([CD25, Propositions C and D]). *(i) If H has polynomial growth of degree $d \geqslant 1$, then*

$$j_{1,\mathsf{Shuffler}^{\circ n}(H)}(x) \simeq \left(\frac{\ln^{\circ n}(x)}{\ln^{\circ(n+1)}(x)}\right)^{\frac{1}{d}}$$

for any $n \geqslant 1$.

(ii) If $j_{1,H}$ satisfies Assumptions $(\star)$ and $(\star\star)$, then

$$j_{1,\mathsf{Shuffler}^{\circ n}(H)}(x) \simeq j_{1,H}(\ln^{\circ n}(x))$$

for any $n \geqslant 1$.

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